

ON CAPACITY OF FREQUENCY SELECTIVE MULTIPATH CHANNELS WITH PATH CORRELATIONS

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ABSTRACT

This paper presents results of ergodic capacity of frequency selective multipath fading channels with path correlations. The exact capacity expression is derived. The main result establishes that, for the uncorrelated case, the ergodic capacity does not depend on either the number of taps or the power delay profile of the channel. However when tap correlation is taken into account, this statement is no longer valid, and high dependence on the channel profile is observed. Although it is impossible to test the impact of different tap correlation models exhaustively, detailed analytical study is carried out under a particular neighbor tap correlation model, i.e., only two neighbor taps are correlated. The worst case that yields the minimal capacity is identified after the intractable and non-convex problem is translated into a convex optimization one. An exact analytical power delay profile for arbitrary number of taps is derived in this case.

Index Terms— Ergodic channel capacity, frequency selective channels, Rayleigh fading, tap correlation profile, power delay profile, convex optimization.

1. INTRODUCTION

Channel capacity over wideband fading channels has received great attention because of the emerging third-generation cellular and ultra-wideband (UWB) technologies. Since Shannon's result for the channel capacity, this topic has been studied and extended in many ways. Kennedy [1] has studied the capacity for dispersive fading channels. Gallager [2] has found the capacity for an additive Gaussian noise channel with input constrained in power and frequency. The channel capacity has been derived for a flat Rayleigh fading channel in [3], and for

a frequency selective channel with two independent taps in [4]. Biglieri [5] has formalized and derived important capacity results regarding fading channels. These pioneering works addressed either a flat channel or multipath channels with uncorrelated paths. However, little attention has been paid to the correlated case for frequency selective long multipath channels, while correlation among the channel taps has been evidenced in many measurement campaigns as in [6, 7], especially in dense multipath environments.

In this paper, the channel capacity over frequency selective multipath channels with correlated paths is investigated. Using a widely adopted tapped delay line model to describe a continuous channel, and assuming that the receiver has perfect channel state information, the ergodic (average) channel capacity is studied. It is also assumed that the channel does not vary too fast when compared to the coherence time of the channel, so the block fading model applies. The result is presented in an integral form as a function of the bandwidth W , the correlation coefficient between the i -th and j -th taps, i.e., ρ_{ij} , and also the power of each tap σ_i^2 . For the uncorrelated case, the general result reduces to a simpler closed-form solution which is not dependent on the number of taps L .

As is shown in this paper, once tap correlation is taken into account, the capacity highly depends on the profile of the channel, namely power delay profile, tap correlation profile, and number of taps. An interesting question would arise as what is the worst case in the sense that the capacity is minimized? In another words, what is the worst power delay profile of the channel taps that yields the minimum channel capacity? It turns out that this optimization problem cannot be solved analytically for an arbitrary tap correlation profile. Assuming correlation only between two neighbor taps, which is a physically reasonable assumption, the equivalent convex problem is then formulated as a quadratic constrained quadratic programming (QCQP) problem. An exact solution is found by the standard Lagrangian multiplier

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method. The optimal power delay profile under this neighbor tap correlation assumption is compared with that under an exponential tap correlation profile. It is found that the difference is insignificant, indicating that the neighbor correlation model well approximates an exponential correlation profile. The analytical result based on the optimal power delay profile under the neighbor correlation model is also compared with a virtually interesting uniform power delay profile. For large number of taps, the gap tends to diminish.

The above result shows that for certain class of channels, the capacity might not reach an expected high value under tap correlation. Thus tap correlation has a pronounced effect on channel capacity. The resulting optimal power delay profile may be viewed to be loosely linked to a non-line of sight (NLOS) scenario where the dominant path arrives at some time offset from the first arrival. If we impose additional constraints on the power delay profile for two consecutive taps $\sigma_i^2 \geq \sigma_{i+1}^2$ ($i = 1, \dots, L-1$), such as that conforms line of sight (LOS) channels, the result approaches an almost "uniform" solution, as will be seen later.

2. CHANNEL MODEL

When the transmitted signal $X(f)$ has a bandwidth W greater than the coherence bandwidth B_{coh} of the channel, the frequency components of $X(f)$ with frequency separation exceeding B_{coh} are subject to different attenuations. In such a case, the channel is said to be frequency-selective. When $W > B_{coh}$, the multipath components in the channel response that are separated in delay by at least $1/W$ are resolvable. In this case, the sampling theorem may be used to represent the resolvable received signal components. As shown in [8], the time-invariant frequency-selective channel can be modeled as a tapped delay line with tap spacing $1/W$ as [5,7], $h(t) = \sum_{i=1}^L \alpha_i \delta\left(t - \frac{(i-1)}{W}\right)$, where the tap coefficient is given as $\alpha_i = x_i + j y_i$ with $j = \sqrt{-1}$, and the real part x_i and imaginary part y_i are independent and identically distributed (i.i.d.) Gaussian random variates with zero mean and the same variance $\sigma_i^2/2$, and L is the number of taps. Then α_i has variance σ_i^2 . Assume the correlation coefficient of x_i and x_k is the same as that of y_i and y_k , denoted as ρ_{ik} . Since $\mathbb{E}[x_i y_k] = 0$ for all i and k ($\mathbb{E}[\cdot]$ stands for expectation), it can be easily found that the correlation coefficient of α_i and α_k , $\frac{\mathbb{E}[\alpha_i \alpha_k^*]}{\sigma_i \sigma_k}$, is also ρ_{ik} . If the transmitted and received signals are denoted by $X(t)$ and $Y(t)$ respectively, the input/output relation becomes

$$Y(t) = \sum_{i=1}^L \alpha_i X\left(t - \frac{(i-1)}{W}\right) + N(t), \quad (1)$$

where $N(t)$ is zero-mean white Gaussian noise with double-sided power spectral density N_0 . The power of the transmitted signal is bounded by P , i.e., $\mathbb{E}(X^2(t)) = \int_{-W/2}^{W/2} S_X(f) df \leq P$, where W is the bandwidth of the channel in Hertz, and S_X denotes the power spectral density of the process $X(t)$. For fair comparisons of different capacity results later on, the channel is normalized to have unit energy, i.e.,

$$\sum_{i=1}^L \sigma_i^2 = 1. \quad (2)$$

The Fourier transform of $h(t)$ directly gives the channel frequency response by

$$H(f) = \mathcal{F}(h(t)) = \sum_{i=1}^L \alpha_i e^{-j 2\pi f (i-1)/W}. \quad (3)$$

Our subsequent developments will primarily rely on (2) and (3).

3. ERGODIC CHANNEL CAPACITY

Consider a uniform power allocation for $S_X(f)$ at the transmitter side, which means that $S_X(f)$ is constant over the bandwidth $-W/2 \leq f \leq W/2$. Define transmitter signal-to-noise ratio (SNR) as $\gamma_t = \frac{P}{W N_0}$. Assume that channel information is available only at the receiver side but not at the transmitter side. Using the mutual information definition in nats given in [2, 5], $I = \int_{-W/2}^{W/2} \ln\left(1 + \gamma_t |H(f)|^2\right) df$. The ergodic channel capacity is given by the expected value of I with respect to the fading channel coefficients as

$$C = \int_{-W/2}^{W/2} \mathbb{E}\left[\ln\left(1 + \gamma_t |H(f)|^2\right)\right] df. \quad (4)$$

Since all α_i 's are zero-mean complex Gaussian random variables, $H(f)$ by (3) is itself a zero-mean complex Gaussian random variable for any given f . Applying (2), its variance can be found to be

$$\sigma_H^2(f) = 1 + \sum_{\substack{i,k=1 \\ i \neq k}}^L \sigma_i \sigma_k \rho_{ik} \cos\left(\frac{2\pi f (i-k)}{W}\right), \quad (5)$$

and is frequency dependent. Note that the correlation coefficients appear only in the variance of the distribution. Define $\beta = |H(f)|^2$ that appears in (4). Note that β is the modulus squared of a Gaussian distributed variable. Therefore it is an exponential distributed random variable with mean $\sigma_H^2(f)$ and probability density function as $p(\beta, f) = \frac{1}{\sigma_H^2(f)} e^{-\frac{\beta}{\sigma_H^2(f)}}$. Evaluating the expected value and using the exponential distribution, (4)

becomes

$$C = \int_{-W/2}^{W/2} \Gamma\left(0, \frac{1}{\gamma_t \sigma_H^2(f)}\right) e^{-\frac{1}{\gamma_t \sigma_H^2(f)}} df, \quad (6)$$

where $\Gamma(\cdot, \cdot)$ is the incomplete Gamma function defined as $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$. For given system parameters and channel tap correlation profile, the capacity can be numerically evaluated. Unfortunately, there is no closed-form solution to (6) if no further assumption is made. But for some special cases of particular interest, some insights into capacity can be obtained as follows.

3.1. Uncorrelated Tap Coefficients

The simplest case is when channel tap coefficients are uncorrelated ($\rho_{ij} = 0, i \neq j$). In this case, $\sigma_H^2(f) = 1$ according to (5). Therefore, (6) becomes $C = W \Gamma(0, \gamma_t^{-1}) e^{\gamma_t^{-1}}$. This expression has exactly the same form as the flat Rayleigh fading case given in [3, eq. (5)]. That means, the ergodic capacity over a frequency-selective multipath fading channel with independent taps is the same as that of a flat-fading channel. It does not depend on the number of taps, but only on SNR and channel bandwidth.

4. CAPACITY UNDER A NEIGHBOR TAP CORRELATION MODEL

The expression given in (6) is very general and takes into account all possible kinds of correlations between L taps of the channel. The aim of this section is to gain some insights about the power delay profile which yields the minimum capacity. A neighbor tap correlation model will be introduced. This model will lead to reduced mathematical complexity and provide a trackable solution. Although simple and in closed form, the optimal solution well approximates that under an exponential tap correlation profile. It also shows meaningful relations to a uniform power delay profile.

In the neighbor tap correlation model, it is assumed that correlation is non-zero between two neighbors and zero otherwise. Furthermore, the correlation is the same for all the neighbors, i.e., $\rho_{ij} = \rho, |i - j| = 1, \rho_{ij} = 0, |i - j| > 1$. With this definition, $\sigma_H^2(f)$ in (5) reduces to

$$\sigma_H^2(f) = 1 + 2\rho \cos\left(\frac{2\pi f}{W}\right) \sum_{i=1}^{L-1} \sigma_i \sigma_{i+1}. \quad (7)$$

The capacity is obtained by substituting (7) in (6). Since (7) is the variance of (3), it must be a positive quantity for any f . Applying $f = W/2$, then it follows that

$$\rho < \frac{1}{2 \sum_{i=1}^{L-1} \sigma_i \sigma_{i+1}}. \quad (8)$$

From the Extreme Value Theorem, which says that, in a closed interval, a continuous function must attain its

maximum and minimum values, each at least once. An interesting problem now is to find the vector $\boldsymbol{\sigma} = [\sigma_1, \dots, \sigma_L]^T \in \mathbf{R}_+^L$ which minimizes the capacity, where the operator $(\cdot)^T$ stands for the transpose. As we know, the maximum of the capacity is achieved for the flat fading case, i.e., only one tap ($L = 1$). Finding the worst case that gives the minimum capacity is more difficult. Treating $\boldsymbol{\sigma}$ as a vector variable and observing the result on the capacity, the following optimization problem can be formulated

$$\min_{\boldsymbol{\sigma}} C(\boldsymbol{\sigma}), \quad \text{s.t.} \quad \sum_{i=1}^L \sigma_i^2 = 1. \quad (9)$$

Unfortunately, $C(\boldsymbol{\sigma})$ is not convex over $\boldsymbol{\sigma}$. But an equivalent convex optimization problem will be found.

Lemma 1: Noticing the positiveness of $\gamma_t \sigma_H^2(f)$, the following holds

$$\begin{aligned} & \arg \min_{\boldsymbol{\sigma}} \int_{-W/2}^{W/2} \Gamma\left(0, \frac{1}{\gamma_t \sigma_H^2(f)}\right) e^{-\frac{1}{\gamma_t \sigma_H^2(f)}} df \\ &= \arg \max_{\boldsymbol{\sigma}} \int_{-W/2}^{W/2} \frac{1}{\gamma_t \sigma_H^2(f)} df \end{aligned} \quad (10)$$

where the function $\arg \max_x \Psi(x)$ is the value of x for which $\Psi(x)$ reaches the maximum.

Proof: Notice that for monotonically decreasing positive function $\Psi(\cdot)$ and positive $x(\cdot)$, the following is true $\arg \min_{\boldsymbol{\sigma}} \Psi(x(\boldsymbol{\sigma})) = \arg \max_{\boldsymbol{\sigma}} x(\boldsymbol{\sigma})$. It is clear that γ_t is always positive and the variance $\sigma_H^2(f)$ of $H(f)$ is always positive. Also the function $\Gamma(0, x) e^x$ is monotonically decreasing and positive for positive x . Treating $\frac{1}{\gamma_t \sigma_H^2(f)}$ as $x(\cdot)$ and noticing that integration preserves the convexity, lemma is proved. \square

Under this lemma, the optimization problem by (9) is equivalent to the following one after suppressing constant γ_t

$$\max_{\boldsymbol{\sigma}} \int_{-W/2}^{W/2} \frac{1}{\sigma_H^2(f)} df, \quad \text{s.t.} \quad \sum_{i=1}^L \sigma_i^2 = 1. \quad (11)$$

Fortunately, for $\sigma_H^2(f)$ defined in (7), the integral above has a closed form expression

$$\int_{-W/2}^{W/2} \frac{1}{\sigma_H^2(f)} df = \frac{W}{\sqrt{1 - \left(2\rho \sum_{i=1}^{L-1} \sigma_i \sigma_{i+1}\right)^2}}. \quad (12)$$

It is observed again that the condition of (8) should be satisfied for the above result to be meaningful. Therefore, the optimization problem of (11) and thus the original problem (9) is translated into the maximum of (12) with variable $\boldsymbol{\sigma}$. To conclude, the maximum of (12) is found to be the equivalent convex problem as

$$\max_{\boldsymbol{\sigma}} \sum_{i=1}^{L-1} \sigma_i \sigma_{i+1}, \quad \text{s.t.} \quad \sum_{i=1}^L \sigma_i^2 = 1. \quad (13)$$

The objective function in (13) can be written in a matrix form as $\boldsymbol{\sigma}^T \mathbf{A} \boldsymbol{\sigma}$, where \mathbf{A} is an $L \times L$ positive-semidefinite matrix with zeros everywhere except in the off diagonal $a_{i,i+1} = 1$, $i = 1 \dots L - 1$.

Therefore, (13) becomes a quadratically constrained quadratic program problem (QCQP)

$$\max_{\boldsymbol{\sigma}} \boldsymbol{\sigma}^T \mathbf{A} \boldsymbol{\sigma}, \quad \text{s.t.} \quad \|\boldsymbol{\sigma}\|^2 = 1. \quad (14)$$

In order to find its solution, the Lagrangian function can be written as $\mathcal{L}(\boldsymbol{\sigma}, \lambda) = \sum_{i=1}^{L-1} \sigma_i \sigma_{i+1} + \lambda \left(1 - \sum_{i=1}^L \sigma_i^2\right)$, with variables σ_i and multiplier λ . Setting the derivative of \mathcal{L} with respect to σ_i for all i to zero, the following is attained

$$\frac{d\mathcal{L}(\boldsymbol{\sigma}, \lambda)}{d\sigma_i} = \begin{cases} \sigma_2 - 2\lambda\sigma_1 = 0, & i = 1 \\ \sigma_{i+1} - 2\lambda\sigma_i + \sigma_{i-1} = 0, & i = 2, \dots, L-1 \\ \sigma_{L-1} - 2\lambda\sigma_L = 0, & i = L \end{cases} \quad (15)$$

Now we turn our attention to finding both λ and all σ_i . From (15), we obtain $\lambda = \frac{\sigma_2}{2\sigma_1} = \frac{\sigma_{i+1} + \sigma_{i-1}}{2\sigma_i} = \frac{\sigma_{L-1}}{2\sigma_L}$. Since $a/b = c/d = (a+c)/(b+d)$, then $\lambda = \frac{\sum_{i=1}^{L-1} \sigma_i + \sum_{i=2}^L \sigma_i}{2 \sum_{i=1}^L \sigma_i} = 1 - \frac{\sigma_1 + \sigma_L}{2 \sum_{i=1}^L \sigma_i}$. Furthermore, it can be shown that $\sigma_i = \sigma_{L-i+1}$, therefore it becomes $\lambda = 1 - \frac{\sigma_1}{\sum_{i=1}^L \sigma_i}$. Note that $0 < \lambda < 1$ (λ can not be zero or one. Otherwise, all σ_i must be zero). All σ_i satisfy a linear constant-coefficient difference equation in (15) and a constraint in (14) is imposed for one additional equation. A closed form solution to λ and all σ_i is found by a characteristic polynomial based direct method as (for the sake of space this proof has been omitted)

$$\lambda = \cos\left(\frac{\pi}{L+1}\right), \quad (16)$$

$$\sigma_i = \sqrt{\frac{2}{L+1}} \sin\left(\frac{i\pi}{L+1}\right). \quad (17)$$

and the corresponding maximum of $\sum_{i=1}^{L-1} \sigma_i \sigma_{i+1}$ is given by $\max_{\boldsymbol{\sigma}} \sum_{i=1}^{L-1} \sigma_i \sigma_{i+1} = \cos\left(\frac{\pi}{L+1}\right)$. With this value in mind, we can use this maximum in (8) to obtain the constraint given by $\rho < (2 \cos(\pi/(L+1)))^{-1}$.

Fig. 1 plots the distribution of optimal σ_i for $L = 20$ by a solid line based on (17). For convenience, we term it as power delay profile although σ_i is the tap standard deviation. For comparison, we also consider a general model including correlation between all the taps. For this aim, we define the exponential tap correlation model as $\rho_{ij} = e^{-\delta|i-j|}$, $\delta > 0$ in (5). For this model and using $L = 20$, $\delta = 0.7$ meets the requirements for $\rho < (2 \cos(\pi/(L+1)))^{-1}$. Using numerical tools to find the minimum of (6), the solution for $\boldsymbol{\sigma}$ that minimizes the capacity under an exponential tap correlation model is

also plotted in a dashed line. Note that the two curves are almost indistinguishable, meaning that the neighbor tap correlation model approximates the exponential tap correlation model very well since the latter used in this figure provides weak influence for distant taps and strong influence for neighbor taps. Under the same parameter settings, i.e. $\delta = 0.7$ and $L = 20$, the channel capacity is plotted as a function of SNR in Fig. 2. In addition to the neighbor and exponential correlation models, the result for AWGN channel is also shown.

5. A CONSTRAINED CHANNEL PROFILE

If no constraint on the power delay profile is imposed except the total normalized power, then the optimal (worst case) profile is similar to an NLOS channel scenario, as observed in Fig. 1. For another class of channels such as LOS channels, one may consider additional constraint as follows $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_L$. The problem that might be interesting is how the optimal power delay profile and minimum capacity are different from the previous case.

The convex problem is now formulated as

$$\max_{\boldsymbol{\sigma}} \sum_{i=1}^{L-1} \sigma_i \sigma_{i+1}, \quad \text{s.t.} \quad \sum_{i=1}^L \sigma_i^2 = 1 \quad \sigma_i \geq \sigma_{i+1} \quad i = 1, \dots, L-1 \quad (18)$$

Introducing two multipliers $\lambda > 0$ and $\boldsymbol{\mu} = [\mu_1, \dots, \mu_{L-1}]^T \in \mathbf{R}_+^{L-1}$, the Lagrangian objective function for this constrained optimization problem can be written as

$$\mathcal{L}(\boldsymbol{\sigma}, \lambda, \boldsymbol{\mu}) = \sum_{i=1}^{L-1} \sigma_i \sigma_{i+1} + \lambda \left(1 - \sum_{i=1}^L \sigma_i^2\right) + \sum_{i=1}^{L-1} \mu_i (\sigma_i - \sigma_{i+1}) \quad (19)$$

Setting the derivative of \mathcal{L} with respect to σ_i to zero, and imposing the KKT conditions $\mu_i (\sigma_{i+1} - \sigma_i) = 0$, the solution can be found by conventional convex optimization techniques. Fig. 3 shows the optimal power delay profiles for both neighbor tap correlation model and exponential correlation model ($\delta = 0.7$) when $L = 20$. The difference is negligible. The exponential tap correlation model $\rho_{ij} = e^{-\delta|i-j|}$, with varying parameter δ , is adopted. In Fig. 4, the channel capacity is shown for two different values of δ , i.e., $\delta = 0.1$ and $\delta = 0.01$, and for four different power delay profile models (exponential $\rightarrow \sigma_i^2 = e^{(-i\Delta)}$, linear-exponential $\rightarrow \sigma_i^2 = i e^{(-i\Delta)}$, uniform profiles, and optimal profile (17) under the exponential correlation model), $\Delta = 0.4$ for linear-exponential and exponential power profile model. For a fixed SNR and given power delay profile, the increased correlation (decreased δ) decreases the capacity. The influence of the tap correlation profile is the most pronounced for the optimal power profile under an exponential correlation model, and decays in the order of uniform, linear-exponential, and exponential power delay profile models. For example, for a SNR of 10dB, the optimal delay profile provides

a channel capacity about 67% less than the exponential model.

6. CONCLUSIONS

This paper studied the ergodic channel capacity for frequency selective channels with path correlations. For the uncorrelated case, it was shown that the number of paths L has no influence on the channel capacity, with corresponding results coincident with the flat fading case, although one can expect an increased complexity at the receiver side. For the correlated case, the capacity highly depends on the channel profile. The tap correlation, power delay profile and number of taps all contribute to the capacity result.

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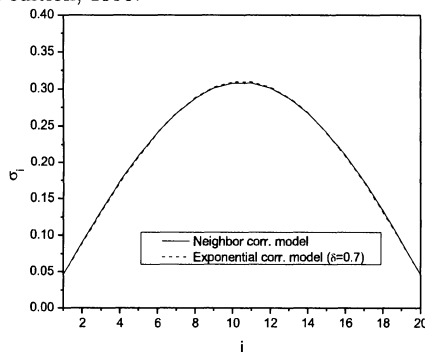


Fig. 1. Optimal power delay profiles that yield the minimum channel capacity for two tap correlation models.

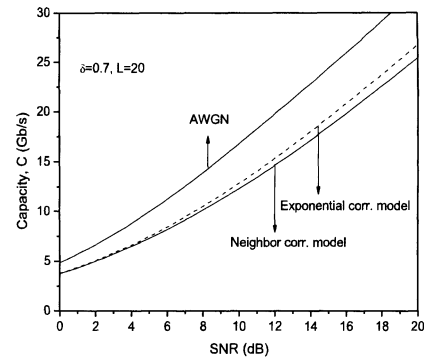


Fig. 2. Comparison between the optimal power delay profile for exponential correlation and neighbor correlation models.

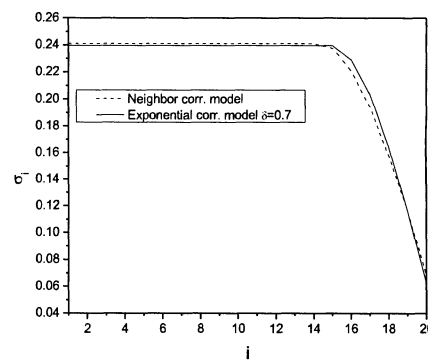


Fig. 3. Optimal power delay profile when additional constraints are imposed.

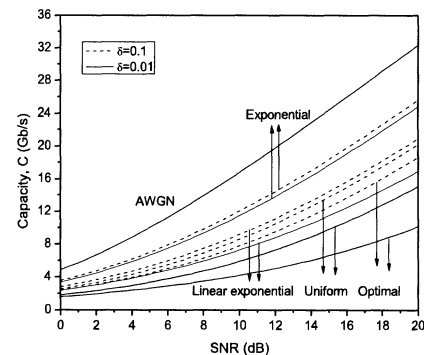


Fig. 4. Channel capacity for different power delay profiles and tap correlations, $L = 20$.